Solution by W. D. LAMBERT, Washington, D. C.

Let r be the variable radius of the sphere, while forming; a the final radius $=\dfrac{6.370\times10^8}{49}$ centimeters; ρ the density of water$=1$; k the gravitation constant$=6.665\times10^{-8}$ dynes; J the mechanical equivalent of heat$=4.184\times10^7$ ergs for the centigrade gram-calorie. It is a little easier to conceive the sphere as pulled asunder against its own attraction, and the amount of work will be the same. Suppose the sphere made up of layers, each of thickness dr, and that the sphere has been reduced to radius r. The mass of a layer is $4\pi\rho r^2dr$. The attraction between this mass and the remaining sphere is $\dfrac{k\times\frac{4}{3}\pi\rho r^3\times4\pi\rho r^2dr}{x^2}$, where x denotes the distance of the layer (supposed to be scattered symmetrically) from the center of the sphere. The work done in removing the layer from the surface of the sphere in question to infinity is $\frac{16}{3}\pi^2k\rho^2r^5dr\int_r^\infty\dfrac{dx}{x^2}=\frac{16}{3}\pi^2k\rho^2r^4$.

The total work done in removing all layers is

$$\tfrac{16}{3}\pi^2k\rho^2\int_0^a r^4dr=\tfrac{16}{15}\pi^2k\rho^2a^5.$$

Dividing this quantity by the mass and by J we get for the temperature $\frac{4}{5}\dfrac{\pi k\rho a^2}{J}$. For substances other than water this result should be multiplied by the specific heat of the substance. Using the numerical values previously given, we get for the temperature $0^\circ.677$ centigrade.

Also solved by G. B. M. Zerr, whose result is 0.656. This difference of result is due to the different values assumed for the constants entering into the solution.

AVERAGE AND PROBABILITY.

183. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A point within a given triangle is joined to each of the corners. What is the average of the sum of the lengths of these three lines?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, P the random point, A the vertex, BC the base of the triangle, AD the altitude. Through P draw QR, parallel to BC cutting AD in F. Let $AD=p$, $BD=e$, $DC=d$, $AF=x$, $FP=y$. Then $AP=\sqrt{[x^2+y^2]}$. The limits of x are 0 and p; of y, $-QF=ex/p$ and $+FR=dx/p$. Let M=average length of AP, Δ=average length of the sum.

$$\therefore M=\int_0^p\int_{-ex/p}^{dx/p}\sqrt{[x^2+y^2]}\,dxdy\Big/\int_0^p\int_{-ex/p}^{dx/p}dxdy$$

$$=\frac{2}{ap}\int_0^p \int_{-ex/p}^{dx/p} \surd[x^2+y^2]dxdy, \text{ since } d+e=a,$$

$$=\frac{1}{ap}\int_0^p \left[\frac{[bd+ce]x^2}{p^2}+x^2\log\left(\frac{d+b}{c-e}\right)\right]dx, \ \left(\begin{matrix}p^2+d^2=b^2\\ p^2+e^2=c^2\end{matrix}\right.$$

$$=\frac{1}{3a}\left[bd+ce+p^2\log\left(\frac{d+b}{c-e}\right)\right]$$

$$=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+b^2\sin^2 C\log\left(\frac{b[1+\cos C]}{b[1-\cos B]}\right)\right].$$

By similarity,

$$\varDelta=\frac{1}{3a}\left[b^2\cos C+c^2\cos B+\frac{4\varDelta^2}{a^2}\log\left(\frac{a+b+c}{b+c-a}\right)\right]+\frac{1}{3b}\left[c^2\cos A+a^2\cos C\right.$$

$$\left.+\frac{4\varDelta^2}{b^2}\log\left(\frac{a+b+c}{a+c-b}\right)\right]+\frac{1}{3c}\left(a^2\cos B+b^2\cos A+\frac{4\varDelta^2}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right],$$

where $\varDelta$=area of triangle.

$$\therefore \varDelta=\tfrac{1}{3}[a+b+c]+\frac{1}{6a^2}[b+c][b-c]^2+\frac{1}{6b^2}[a+c][a-c]^2$$

$$+\frac{1}{6c^2}[a+b][a-b]^2+\frac{4\varDelta^2}{3}\left[\frac{1}{a^3}\log\left(\frac{a+b+c}{b+c-a}\right)+\frac{1}{b^3}\log\left(\frac{a+b+c}{a+c-b}\right)\right.$$

$$\left.+\frac{1}{c^2}\log\left(\frac{a+b+c}{a+b-c}\right)\right].$$

If $a=b=c$, $\varDelta=a[1+\frac{3}{4}\log 3]$.

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be the point, and $AD=h$, the perpendicular from A upon BC. Put $AP=x$, and $\angle PAD=\theta$. Then the average length of AP is

$$\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta} x^2d\theta dx \div \int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\int_0^{h\sec\theta} xd\theta dx=\frac{2h}{3}\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^3\theta d\theta\div\int_{B-\frac{1}{2}\pi}^{\frac{1}{2}\pi-C}\sec^2\theta d\theta$$

$$=\frac{h}{3}\left(\cot C\operatorname{cosec} C+\cot B\operatorname{cosec} B-\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B]\div\cot C+\cot B\right.$$

$$=\frac{1}{3a}\left(b^2\cos C+c^2\cos B-bc\sin B\sin C\log[\tan\tfrac{1}{2}C\tan\tfrac{1}{2}B]\right.$$

In like manner it may be shown that the average length of BP is

$$\frac{1}{3b}\left(a^2\cos C+c^2\cos A-ac\sin A\sin C\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}C]\right.;\text{ and of } CP,$$

$$\frac{1}{3c}\left(a^2\cos B+b^2\cos A-ab\sin A\sin B\log[\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B]\right..$$

Hence the required average is

$$M=\tfrac{1}{3}\Big[\left(\frac{b^2}{c}+\frac{c^2}{b}\right)\cos A+\left(\frac{a^2}{c}+\frac{c^2}{a}\right)\cos B+\left(\frac{a^2}{b}+\frac{b^2}{a}\right)\cos C-a\sin\tfrac{1}{2}A[b\sin B$$

$$+c\sin C]\log\tan\tfrac{1}{2}A-b\sin B[a\sin A+c\sin C]\log\tan\tfrac{1}{2}B$$

$$-c\sin C[a\sin A+b\sin B]\log\tan\tfrac{1}{2}C.$$

184. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of the sides of a given square, straight lines are drawn across the square in every possible direction. What is their average length?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The problem evidently wants the average length of all lines terminated in opposite sides; otherwise the problem is the same as problem 169, three solutions of which have already been published.

Let $[a, x]$ be the coordinates of one end of the line, $[0, y]$ the coordinates of the other end. $\triangle$ =the average length required.

$$\therefore \triangle=\frac{\int_0^a\int_0^x \sqrt{\{a^2+[x-y]^2\}}dx\,dy}{\int_0^a\int_0^x dxdy},=\frac{2}{a^2}\int_0^a\int_0^x\sqrt{\{a^2+[x-y]^2\}}dxdy$$

$$=\frac{1}{a^2}\int_0^a\left(x\sqrt{[a^2+x^2]}+a^2\log\frac{x+\sqrt{[a^2+x^2]}}{a}\right)dx=a\{\tfrac{2}{3}[1-\sqrt{2}]+\log[1+\sqrt{2}]\}.$$

II. Solution by HENRY HEATON, Belfield, N. D.

Let P be a point in AB, PE a line perpendicular to AB, and PF a line drawn across the $\triangle PAD$. Put $AP=x$, and $\angle FPA=\theta$. Then supposing x constant, the average length of the lines drawn from P across the triangle PAD is

$$\int_0^{\tan^{-1}(a/x)}x\sec\theta\,d\theta\div\int_0^{\tan^{-1}(a/x)}d\theta=x\log\left(\frac{\sqrt{[a^2+x^2]}+a}{x}\right)\div\tan^{-1}[a/x].$$